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Supersymmetric index and s-rule for type IIB branes

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Abstract: We investigate the supersymmetric index of $N = 2, 3$ SU(n) supersymmetric Yang-Mills Chern Simons theories at level k by using the brane configuration with a (p, q)5-brane. We can explain that the supersymmetry breaking occurs when $k < n$ in terms of the s-rule for type IIB branes. The supersymmetric index coincides with the number of the possible supersymmetric brane configurations. We also discuss a construction of a family of theories which have the same supersymmetric index.

Keywords: Chern-Simons Theories, Brane Dynamics in Gauge Theories, Supersymmetry Breaking.

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Contents

1. Introduction

A supersymmetric index (Witten index) $\mathrm{Tr}(-1)^F$ is a useful criterion of dynamical supersymmetry breaking in supersymmetric field theories [1]. The index is given by a difference of the number of bosonic and fermionic zero energy states and it counts the number of the supersymmetric vacua. When $\mathrm{Tr}(-1)^F \neq 0$ supersymmetry is not spontaneously broken since if pairs of bosonic and fermionic zero energy states gains a non-zero energy at least $\mathrm{Tr}(-1)^F$ states remain as zero energy states. Conversely, however, for the case of $\mathrm{Tr}(-1)^F = 0$ we can not conclude whether supersymmetry is broken or not. But it indicates a possibility of spontaneous supersymmetry breaking.

The indexes for various supersymmetric theories have been computed. Recently, the index for supersymmetric gauge theory with a mass gap, which is $N = 1$ supersymmetric Yang-Mills Chern-Simons theory with gauge group G, is given by Witten [2]. His result shows that $\mathrm{Tr}(-1)^F \neq 0$ if $|k| \geq h/2$, where h is the dual Coxeter number of G, and suggests that dynamical symmetry breaking occurs for $|k| < h/2$.

Three-dimensional supersymmetric Yang-Mills Chern-Simons theory can be realized on the brane configuration with a (p, q)5-brane in type IIB superstring theory [3, 4, 5, 6]. For $N = 1$ supersymmetric theory without any extra matters, the corresponding brane configuration has not been well understood. But $N = 2, 3$ supersymmetric theories are easy to handle by the branes. The moduli space of vacua of $N = 2, 3$ theory can be explained by the brane configuration [4]. So we focus only $N = 2, 3$ theory throughout the present paper.

In this paper, we extend the Witten's computation of the index to $N = 2, 3$ SU(n) super Yang-Mills Chern-Simons theory at level k and show that this index is well interpreted by using the type IIB brane configuration. In particular, the dynamical supersymmetry breaking in some region of k is explained by the so-called "s-rule" (supersymmetric rule) for the branes. Moreover, when the index is non-zero we show that the index exactly coincides of the number of possible supersymmetric brane configurations.

The organization of this paper is as follows. In section 2, we briefly review a microscopic computation of the index by Witten. The computation can straightforwardly extend to $N = 2, 3$ extended supersymmetry. We will give the formulae of the indexes. In section 3, we explain about the s-rule for the branes and generalize it to the stretched D3-branes between two different types of (p, q)5-branes. We next show in section 4 that if this generalized s-rule is applied to the $N = 2, 3$ Yang-Mills Chern-Simons brane configuration, the configuration for $k < n$ can not keep the s-rule, that is, the configuration is not supersymmetric. These results are consistent with the computation of the index. We will give the exact formula of the index by counting the number of possible supersymmetric configuration from M-theoretical point of view. We also discuss the relation between a supersymmetric quantum mechanics and the computation of the index in terms of the branes. In the subsequent subsection, we construct a family of theories which have the same index by using some brane dynamics. Finally, section 5 is devoted to a summary of our results and discussions of further problems.

2. Brief review of the microscopic computation of the index

We first consider $N = 1$ SU(n) supersymmetric Yang-Mills Chern-Simons theory on $\mathbb{R} \times T^2$. The computation is done by a Born-Oppenheimer approximation. In this approximation we take a small volume limit of the torus. If g and r denote the gauge coupling and the radius of the torus, the mass of the vector multiplet kg^2 is much smaller than the Kaluza-Klein mass of order $1/r$. We will consider the quantizing of the "zero energy" states, whose energy are of order kg^2 by means of the approximation.

Now let us consider the moduli space of a flat G-connection on T^2, which is a zero energy classical gauge field configuration. The moduli space $\mathcal{M}$ of flat G-connections on T^2 is given by

$$\mathcal{M} = (\mathbf{U} \times \mathbf{U})/W, \tag{2.1}$$

where $\mathbf{U}$ is the maximal torus of G and W is the Weyl group. For $G = \mathrm{SU}(n)$, $\mathcal{M}$ becomes simply the complex projective space $\mathbb{CP}^{n-1}$ [7].

We next consider "zero modes" for the gluino fields of positive and negative chirality λ_+ and λ_-, which have at most energy of order kg^2. We assume the zero modes of $\lambda_\pm$ take of the form:

$$\lambda_\pm = \sum_{a=1}^{r} \eta_\pm^a T^a \,, \tag{2.2}$$

where the $T^a, a = 1, \ldots, r$ are a basis of the Lie algebra of $\mathbf{U}$ and $\eta_\pm^a$ are fermionic constants, which obey the following canonical anti-commutation relations

$$\{\eta_+^a, \eta_-^b\} = \delta^{ab} \,, \qquad \{\eta_+, \eta_+\} = \{\eta_-, \eta_-\} = 0 \,. \tag{2.3}$$

The η_+ and η_- can be regarded as creation and annihilation operators, so we now introduce ground states annihilated by the $\eta_\pm^a$

$$\eta_\pm |\Omega_\pm\rangle = 0 \,. \tag{2.4}$$

These two states are related each other by

$$|\Omega_+\rangle = \prod_{a=1}^{r} \eta_+^a |\Omega_-\rangle \,. \tag{2.5}$$

The anti-commutation relations (2.3) can be regarded as the Clifford algebra on $\mathcal{M}$. Therefore, the Hilbert space made by quantizing the fermion zero modes maps to the space of spinor fields on $\mathcal{M}$. Since $\mathcal{M}$ is a complex manifold, the spinor field on $\mathcal{M}$ is simply represented by the form with values in $K^{1/2}$, where K is the canonical line bundle of $\mathcal{M}$. So if we consider a general state in the fermionic space

$$\eta_-^{a_1} \cdots \eta_-^{a_q} |\Omega_+\rangle \,, \tag{2.6}$$

this is regarded as the $(0, q)$-form on $\mathcal{M}$ with values in $K^{1/2}$. More precisely speaking, the forms on $\mathcal{M}$ take values not only in $K^{1/2}$ but also in a line bundle $\mathcal{W}$, since $|\Omega_+\rangle$ is regarded as a $(0, 0)$-form on $\mathcal{M}$ with values in $\mathcal{W}$.

To determine the line bundle $\mathcal{W}$ let us consider the canonical quantization of the Yang-Mills Chern-Simons theory. The momentum conjugate to A_i^a is given by

$$\Pi_i^a = \frac{1}{g^2} F_{0i}^a - \frac{k}{4\pi} \epsilon_{ij} A_j^a \,. \tag{2.7}$$

Writing formally $\Pi_i^a = -i\delta/\delta A_i^a$, we have

$$\frac{1}{g^2} F_{0i}^a = -i \frac{D}{DA_i^a} \,, \tag{2.8}$$

where

$$\frac{D}{DA_i^a} = \frac{\delta}{\delta A_i^a} + i \frac{k}{4\pi} \epsilon_{ij} A_j^a \tag{2.9}$$

is a connection on the line bundle $\mathcal{W}$. The connection form $\frac{k}{4\pi}\epsilon_{ij}A_j^a$ means that the line bundle over $\mathcal{M}$ is $\mathcal{L}^k$, where $\mathcal{L}$ is the basic line bundle. The supercharges of the theory are written in terms of the connection

$$Q_+ = \frac{1}{g^2}\int_{T^2} \mathrm{Tr} F_{0z}\lambda_- = \int_{T^2} \mathrm{Tr}\lambda_- \frac{D}{DA_{\bar{z}}}\,, \tag{2.10}$$

$$Q_- = \frac{1}{g^2}\int_{T^2} \mathrm{Tr} F_{0\bar{z}}\lambda_+ = \int_{T^2} \mathrm{Tr}\lambda_+ \frac{D}{DA_z}\,. \tag{2.11}$$

Therefore, the supercharges Q_+ and Q_- can be identified with the $\bar{\partial}$ and $\bar{\partial}^\dagger$ operators, which is a decomposition of a Dirac operator acting on spinors valued in $\mathcal{W} = \mathcal{L}^k$.

Since these two operators obey

$$\{\bar{\partial}, \bar{\partial}^\dagger\} = H\,, \qquad \bar{\partial}^2 = (\bar{\partial}^\dagger)^2 = 0\,, \tag{2.12}$$

where H is the hamiltonian, the space of supersymmetric ground states is given by the cohomology

$$\bigoplus_{i=0}^{n-1} H^i\left(\mathcal{M}, \mathcal{W} \otimes K^{1/2}\right). \tag{2.13}$$

For $G = \mathrm{SU}(n)$, $\mathcal{M} \simeq \mathbb{CP}^{n-1}$. The basic line bundle over $\mathcal{M}$ is $\mathcal{L} = \mathcal{O}(1)$ and the canonical bundle of $\mathcal{M}$ is $K = \mathcal{L}^{-n}$. Therefore, the above cohomology groups are rewritten as

$$\bigoplus_{i=0}^{n-1} H^i\left(\mathbb{CP}^{n-1}, \mathcal{L}^{k-n/2}\right), \tag{2.14}$$

where we use $\mathcal{W} = \mathcal{L}^k$. Thus, the supersymmetric index is

$$I_{N=1}(k) = \sum_{i=0}^{n-1}(-1)^i \dim H^i\left(\mathbb{CP}^{n-1}, \mathcal{L}^{k-n/2}\right). \tag{2.15}$$

These cohomology groups are computed by Serre [8] and their dimensions are as follows

$$\dim H^i\left(\mathbb{CP}^{n-1}, \mathcal{L}^r\right) = \begin{cases} 0\,, & \text{for } 0<i<n-1 \\ 0\,, & \text{for } i=n-1,\ r>-n \\ 0\,, & \text{for } i=0,\ r<0 \\ \binom{n+r-1}{r}\,, & \text{for } i=0,\ r\geq 0\,, \end{cases} \tag{2.16}$$

where $\binom{n+r-1}{r}$ is a binomial coefficient and means the dimension of the vector space of degree r homogeneous polynomials in the n homogeneous coordinates of $\mathbb{CP}^{n-1}$. Using this formula we obtain the supersymmetric index of the $N=1$ Yang-Mills Chern-Simons theory,

$$I_{N=1}(k) = \begin{cases} 0\,, & \text{for } |k|<n/2 \\ \binom{k+n/2-1}{k-n/2}\,, & \text{for } |k|\geq n/2\,, \end{cases} \tag{2.17}$$

where for the case of $k \leq -n/2$ we use Serre duality

$$H^{n-1}(\mathbb{CP}^{n-1}, \mathcal{L}^r) \simeq H^0(\mathbb{CP}^{n-1}, \mathcal{L}^{-n-r}), \qquad \text{for } r \leq -n. \tag{2.18}$$

We now extend this computation of the index to the case of $N = 2, 3$ supersymmetric Yang-Mills Chern-Simons theory in three-dimensions. First, we note that the $N = 2$ vector multiplet consists of one spin 1 vector boson A_μ, two spin 1/2 spinors λ_1, λ_2 and one spin 0 real adjoint scalar X. Since the adjoint scalar has the mass kg^2, the Coulomb branch of this theory is completely lifted. Therefore, the presence of the adjoint scalar does not affect the index. So, the computation of the index is modified by two gluino fields as

$$\begin{aligned} I_{N=2}(k) &= \sum_{i=0}^{n-1}(-1)^i \dim H^i\left(\mathbb{CP}^{n-1}, \mathcal{W} \otimes K^{1/2} \otimes K^{1/2}\right), \\ &= \sum_{i=0}^{n-1}(-1)^i \dim H^i\left(\mathbb{CP}^{n-1}, \mathcal{L}^{k-n}\right). \end{aligned} \tag{2.19}$$

Similarly, for the $N = 3$ theory, the massive vector multiplet contains one spin 1 vector boson A_μ, three spin 1/2 spinors $\lambda_1, \lambda_2, \lambda_3$, three spin 0 real adjoint scalars X_1, X_2, X_3 and one spin $-1/2$ spinor χ. In this case, there is also no Coulomb branch moduli. The gluino fields contain one opposite spin field, so the contribution of one of spin 1/2 fields is canceled by the spin $-1/2$ field χ.

$$\begin{aligned} I_{N=3}(k) &= \sum_{i=0}^{n-1}(-1)^i \dim H^i\left(\mathbb{CP}^{n-1}, \mathcal{W} \otimes K^{3/2} \otimes K^{-1/2}\right) \\ &= \sum_{i=0}^{n-1}(-1)^i \dim H^i\left(\mathbb{CP}^{n-1}, \mathcal{L}^{k-n}\right) \end{aligned} \tag{2.20}$$

Therefore, the index of $N = 3$ theory coincides with the $N = 2$ one. We can again obtain the index of the $N = 2, 3$ theories by the Serre's formula.

$$I_{N=2,3}(k) = \begin{cases} 0, & \text{for } 0<k<n \\ \binom{k-1}{k-n} = \binom{k-1}{n-1}, & \text{for } k\geq n. \end{cases} \tag{2.21}$$

For $k \leq 0$ we also compute the index by using the Serre duality, which is non-zero valued for any n, but the field theoretical meaning of this duality has not been understood at the present. So, we assume that k is a positive integer in the following. We will discuss the relation between the index and the brane configuration in the following section.

3. The s-rule for type IIB branes

In this section, we explain about the s-rule for branes in string theory. The s-rule is a phenomenological rule of brane dynamics, which is first proposed by Hanany and Witten [9], and is needed in order to make supersymmetric configurations. For example, a NS5-brane and a D5-brane, which are completely twisted in the configuration space, can be supersymmetrically connected by only one D3-brane. If we use this rule we can find the exact correspondence between the brane configuration and the supersymmetric vacua of field theories. Some explanations on this rule are given from various point of view in refs. [10, 11, 12, 13, 14]. If we map the D3-brane between the NS5-brane and the D5-brane to an M2-brane between two M5-branes by U-duality, we obtain the following rule in M-theory: *a configuration in which two completely twisted M5-brane are connected by more than one M2-brane is not supersymmetric.*

We can obtain all of other supersymmetric brane configurations keeping the $s-$rule in various string theory from this M-theoretical rule by string duality. So we now consider the generalization of the s-rule for the D3-branes between two different type of the (p,q)5-brane. The (p,q)5-brane in type IIB theory is described by a single M5-brane which wrapping simultaneously on two cycles of the compactified torus of M-theory. The wrapping number of the each cycles corresponds to charges of the (p,q)5-brane. This wrapping cycle is denoted by $p\alpha + q\beta$, where α and β stand for the two independent cycles of the torus. If we introduce another (p',q')5-brane the corresponding M5'-brane similarly wraps on a $p'\alpha + q'\beta$ cycle. The M5-brane and the M5'-brane meet $|pq' - qp'|$ times on the torus, where $|pq' - qp'|$ is an intersection number of cycles $p\alpha + q\beta$ and $p'\alpha + q'\beta$. Since only one M2-brane can be attached on each intersecting point if we use the above s-rule in M-theory, the maximal number of the M2-branes between M5- and M5'-brane must be $|pq' - qp'|$ to preserve supersymmetry. If the number of M2-branes is more than $|pq' - qp'|$, we can not arrange the M2-branes in a supersymmetric way.

By using string duality, the M5- and M5'-brane maps to the (p,q)5- and (p',q')5-brane in type IIB theory. So we find that if the number of the D3-branes between the (p,q)5- and (p',q')5-brane is more than $|pq' - qp'|$ then the configuration is not supersymmetric.[1]

4. Comparison with supersymmetric Yang-Mills Chern-Simons theory configuration

In this section, we apply the result of the previous section to the brane configuration of supersymmetric Yang-Mills Chern-Simons theory and compare with the computation of the index. $N = 2,3$ supersymmetric SU(n) Yang-Mills Chern-Simons theories at level k are realized on the generalized Hanany-Witten type configurations, in which n D3-branes are suspended between an NS5-brane and a (p,q)5-brane [3]. The coefficient of the Chern-Simons term, that is, the level of Chern-Simons theory is given

[1] If $pq' - qp'$ is negative, it means that the stretched D3-branes are anti-D3-branes which have opposite orientation and charge.

by $k = p/q$ in this brane setup. For non-abelian gauge theory this coefficient should be an integer due to the quantization condition. So, we set $p = k$ and $q = 1$ in the pages that follow to avoid subtlety.

Moreover, in order to compare with the result of the index computation, we only consider theories without any extra matter except for vector multiplet. For $N = 2$ Yang-Mills Chern-Simons theory, the configuration is[2]

$$\begin{aligned}&\mathrm{NS5}(012345)\,,\\ &\mathrm{D3}(012|6|)\,,\\ &(k,1)5\left(01278\,\left[{5 \atop 9}\right]_{-\theta}\right),\end{aligned} \tag{4.1}$$

and for $N = 3$,

$$\begin{aligned}&\mathrm{NS5}(012345)\,,\\ &\mathrm{D3}(012|6|)\,,\\ &(k,1)5\left(012\,\left[{3 \atop 7}\right]_{\theta}\left[{4 \atop 8}\right]_{\theta}\left[{5 \atop 9}\right]_{-\theta}\right).\end{aligned} \tag{4.2}$$

4.1 Supersymmetric index from branes

We first apply the generalized s-rule to the type IIB brane configuration which describes SU(n) Yang-Mills Chern-Simons theory at level k. In this configuration one of the 5-brane is an NS5-brane, namely, a $(0, 1)$5-brane. Another is a $(k, 1)$5-brane. These 5-branes are completely twisted in the $N = 2$ and $N = 3$ configuration (4.1) and (4.2). So, the supersymmetric configuration is restricted by the s-rule. The intersection number of these 5-branes in M-theory is k. Therefore, for supersymmetry, the maximal number of the D3-branes must be k.

Therefore, we find that the configuration is supersymmetric if $n \leq k$ and the situation for $n > k$ violates the s-rule and spontaneously breaks supersymmetry. This exactly agree with the computation of the index, that is, $I_{N=2,3}(k) \neq 0$ for $n \leq k$ and $I_{N=2,3}(k) = 0$ for $n > k$.

We next consider how the value of the index itself is described in the brane configuration. The value of the index means the number of the possible supersymmetric vacua. So, we can expect that the non-zero value of the index coincides with the number of possible supersymmetric configurations of branes.

In order to count the number of supersymmetric configuration, we again lift the type IIB configuration to M-theory. The NS5-brane is an M5-brane wrapping on a cycle β and the $(k, 1)$5-brane is an M5-brane wrapping on a cycle $k\alpha+\beta$ in M-theory. These M5-branes intersect k times on the torus. Therefore the M2-brane can be attached on k intersecting points, but can not be attached on the same point by the s-rule. If $n > k$, the n M2-branes can not be arranged without violating the s-rule. So, supersymmetry is broken. For $n \leq k$ the number of possible supersymmetric configurations coincides with the number of choices of n points within k points, that is, $\binom{k}{n}$.

[2] For derivation and notation, see ref. [3].

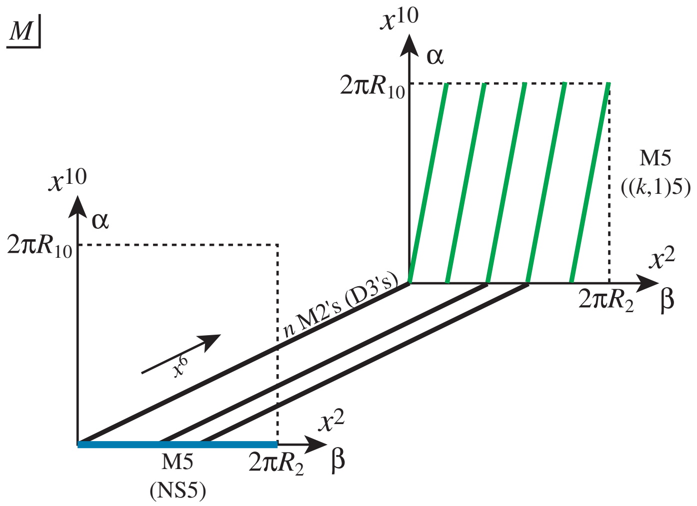

Figure 1: The brane configuration in M-theory for $n \leq k$. n M2-branes can be attached one by one only on k positions.

This number seems not to be the same with the index (2.21). However, the gauge group G on the n D3-branes, which we are considering, is $\mathrm{U}(n)$ rather than $\mathrm{SU}(n)$. This discrepancy comes from the fact that the computation of the index in section 2 is for $G = \mathrm{SU}(n)$. To compare with the index for $\mathrm{SU}(n)$ gauge theory, we must fix one of positions of the M2-brane, which corresponds to a phase of overall $\mathrm{U}(1)$ factor of $\mathrm{U}(n)$. If we fix the position of one M2-brane, residual ways of the supersymmetric arrangement of M2-branes is $\binom{k-1}{n-1}$. This exactly agrees with the index (2.21).

A similar derivation of the supersymmetric index by counting possible supersymmetric configurations is discussed in the case of the M-theory description of $N = 1$ supersymmetric Yang-Mills theory in ref. [15].

As discussed in ref. [3], the positions of M2-branes correspond to vevs of the Wilson line operator W_2 along the x^2-direction. In the supersymmetric configuration, all of M2-brane positions are different. So, we find that the vevs of the Wilson line operator must take the following form

$$\langle W_2 \rangle = \mathrm{diag}\left(e^{2\pi i m_1/k}, e^{2\pi i m_2/k}, \ldots, e^{2\pi i m_n/k}\right), \tag{4.3}$$

up to over all phase, where m_i are different positive integers which satisfy $0 \leq m_1 < m_2 < \cdots < m_n < k$. Therefore, we can conclude that in the supersymmetric phase, gauge symmetry of $N = 2, 3$ $\mathrm{U}(n)$ ($\mathrm{SU}(n)$) Yang-Mills Chern-Simons theory is broken to $\mathrm{U}(1)^n$ ($\mathrm{U}(1)^{n-1}$) by the vev of the Wilson line operator.

4.2 Supersymmetric quantum mechanics on the brane

Let us next consider a dual description of the above type IIB brane configuration which describes supersymmetric Yang-Mills Chern-Simons theory. This picture make easy to learn about the connection to the computation of the index.

We first consider a Maxwell Chern-Simons theory for simplicity. The corresponding brane configuration is that a single D3-brane is suspended between a NS5-brane and a $(k,1)$5-brane. If we consider the long wavelength limit of the Maxwell Chern-Simons lagrangean, in which we drop all spatial derivatives, we obtain

$$L = \frac{1}{2g^2}\dot{A}_i^2 + \frac{k}{2}\epsilon^{ij}\dot{A}_i A_j \,, \tag{4.4}$$

where the gauge coupling g is given by the sting coupling g_s and the difference between two 5-branes L as $1/g^2 = L/g_s$. This lagrangean has exactly the same form as the lagrangean for a non-relativistic charged particle with mass $1/g^2$ moving in the plane in the presence of a external magnetic flux k perpendicular to the plane. Therefore, the Maxwell Chern-Simons system has the same canonical structure with the Landau problem.

On the other hand, if we take T-dual along the x^1- and x^2-direction and S-dual in type IIB theory to the $N=2$ configuration (4.1), we have

$$\begin{aligned}
&\mathrm{D5}(012345)\,,\\
&\mathrm{F1}(0|6|)\,,\\
&k\mathrm{D3}\left(078\left[{}^5_9\right]_{-\theta}\right) \oplus \mathrm{D5}\left(01278\left[{}^5_9\right]_{-\theta}\right),
\end{aligned} \tag{4.5}$$

where kD3$\oplus$D5 is a bound state of k D3-branes and a D5-brane. On the D3-D5 bound state, k D3-branes are regarded as k magnetic flux on the D5-brane. Therefore, on the D3-D5 bound state the end of the fundamental string stretched from the D5-brane looks like a charged particle in the k magnetic flux. (See fig. 2.) Since the gauge field configuration A_i maps to the position of the fundamental string X_i by T-duality, we obtain really the (supersymmetric) quantum mechanical lagrangean of the Landau problem for the position of the string on (x^1, x^2)-plane after using T-duality and S-duality

$$L = \frac{1}{2}\left(\frac{L}{2\pi l_s^2}\right)\dot{X}_i^2 + \frac{\widetilde{k}}{2}\epsilon^{ij}\dot{X}_i X_j \,, \tag{4.6}$$

where l_s is a string length and $\widetilde{k} = k/2\pi l_s^2 g_s$. This is a lagrangean for the particle with mass which is string tension times length L in magnetic flux which is proportional to k as expected from the dualized brane configuration.

If we now extend the above situation to SU(n) gauge theory, n becomes the number of particles. Due to the SU(n) restriction the sum of the particle positions must vanish. The moduli space of the n-tuple of such the particle positions on the torus is known as a copy of complex projective space $\mathbb{CP}^{n-1}$ [7], which is a phase space of the quantum mechanical system (4.6).

After all, analysis of the vacuum structure of supersymmetric Yang-Mills Chern-Simons theory replaces the quantization of the supersymmetric quantum mechanics

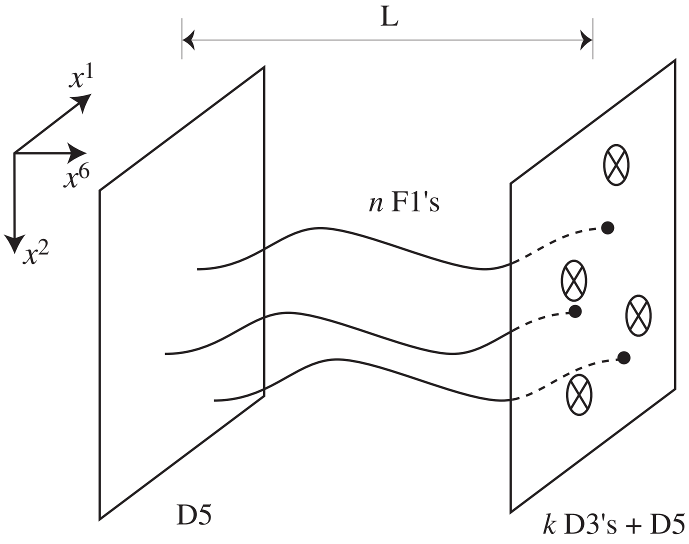

Figure 2: The Landau system on the D3-D5 bound state.

on the D3-D5 bound state by using U-duality. This is the same thing with the microscopic computation of the index in section 2. The covariant derivative (2.9) and the supercharges (2.10) and (2.11) can be regarded as operators acting on the supersymmetric quantum mechanics of the Landau problem.

Finally, we comment on a non-commutative nature on the torus. In limit of $L \ll l_s$, the lagrangean (4.6) becomes $L = \frac{\widetilde{k}}{2}\epsilon^{ij}\dot{X}_i X_j$. This is first order in time derivatives, so the two coordinates X_1 and X_2 are canonically conjugate to one another, that is,

$$[X_i, X_j] = i\frac{\epsilon_{ij}}{\widetilde{k}}\,. \tag{4.7}$$

Thus, the coordinates of the ends of open strings are non-commutative. We can understand this property as the non-commutative torus by turning on a non-zero B-field background using a gauge transformation [16, 17]. This non-commutative nature is closely related to the dynamics of Chern-Simons theory.

4.3 Duality and mirror relations

In this subsection we present a family of theories with a Chern-Simons term, which have the same supersymmetric index. We first consider an exchange of 5-branes in the x^6-direction. When two different types of twisted 5-branes cross in the x^6-coordinate and exchange positions, some D3-branes are annihilated or created by the Hanany-Witten transition [9]. For example, if n D3-branes are stretched between the NS5-brane and the $(k,1)$5-brane ($n < k$), then the number of stretched D3-branes becomes $k-n$ after the transition.

This phenomenon is also simply explained from the M-theoretical point of view [3]. In M-theory, the Hanany-Witten transition is creation or annihilation of a single M2-brane between two twisted M5-branes, which can not avoid each other in the configuration space. In the case of $n < k$, M2-branes can be attached on the n different positions within the k allowed positions on the M5-brane. After exchanging the positions of the two M5-branes in the x^6-coordinate, n M2-branes are disappeared and new M2-branes are created on $k-n$ opening positions.

Thus, we find that by the Hanany-Witten transition the gauge group of $N = 2, 3$ supersymmetric $\mathrm{U}(n)$ Yang-Mills Chern-Simons theory at level k becomes $\mathrm{U}(k-n)$. (If $G = \mathrm{SU}(n)$, then $\hat{G} = \mathrm{SU}(k-n+1)$ since we must fix one of the M2-branes.) On this transition the number of the possible supersymmetric configuration, namely, the index, has not been changed because of the equivalence of the combination $\binom{k}{n} = \binom{k}{k-n}$.

Another operation to the brane configuration is S-duality in type IIB theory. If we apply the S-duality to the brane configuration which describes supersymmetric Yang-Mills Chern-Simons theory at level k, we obtain a self-dual model at level $-1/k$ as a worldvolume effective theory [3]. Since this S-dual transformation can be understood as just a coordinate flip of x^2 and x^{10} in M-theory, there is no difference between the configurations of the Yang-Mills Chern-Simons theory and self-dual model themselves. Therefore, we expect that these theories have the same supersymmetric index.

In this way, we can construct the family of theories with the same index by using the operations in superstring theory. For $N = 2, 3$ supersymmetric theories, $\mathrm{U}(n)$ and $\mathrm{U}(k-n)$ Yang-Mills Chern-Simons theory at level k and $\mathrm{U}(n)$ and $\mathrm{U}(k-n)$ valued self-dual models at level $-1/k$ have all the common supersymmetric index. It suggests that these theories are related each other by duality and mirror symmetry of three-dimensional supersymmetric field theory.

5. Conclusion and discussions

We have investigated in this paper the relation between the supersymmetric index of three-dimensional supersymmetric Yang-Mills Chern-Simons theory and the brane configuration in type IIB superstring theory. We have found that when the index is non-zero, the corresponding brane configuration is supersymmetric. Conversely, when the index vanishes, the configuration violates the s-rule and becomes non-supersymmetric. In addition, we have found that the number of the possible supersymmetric configuration exactly coincides with the value of the index. These results are considered as an explanation of the s-rule for the brane dynamics from the point of view of the worldvolume effective theory.

The computation of the index can be generalized to other gauge groups G of rank r [2]. The moduli space $\mathcal{M}$ of the flat G-connection is a weighted projective space $\mathbb{WCP}^r_{s_0,s_1,\ldots,s_r}$, where the weights $s_0, \ldots, s_r$ are 1 and the coefficients of the highest coroot of G and obey $\sum_{i=0}^r s_i = h$. The supersymmetric condition is also generalized by using this dual Coxeter number h. The corresponding brane configuration should be constructed by adding orientifold planes as like as supersymmetric Yang-Mills theory with orthogonal and symplectic gauge groups [18, 19, 20]. It would be interesting to find the correspondence between the index and the more general s-rule including the orientifold planes.

The original computation of the index is for $N = 1$ Yang-Mills Chern-Simons theory, which is chiral. It is generally hard to construct the chiral theory on the branes. Moreover, the Chern-Simons coefficient of $N = 1$ supersymmetric theory is renormalized and shifted by $-\mathrm{sgn}(k)\frac{h}{2}$ [21]. This renormalization is closely related to the derivation of the index. We have not understood, however, the meaning of the renormalization of the coefficient in terms of the branes in superstring theory. We hope that we can also analyze the dynamics of lower supersymmetric theory by using branes.

The construction of the dual or mirror theory in terms of the brane dynamics is simple as mentioned. However, field theoretical meanings of these symmetries have not been so clear. It is interesting to extend the analysis of ref. [22] to the non-abelian case. We hope that the brane configurations in superstring theory help to understand non-perturbative dynamics of supersymmetric quantum field theories in three-dimensions.

Note added As I completed this paper, I received ref. [23], which also considers supersymmetry breaking in $N = 3$ Yang-Mills Chern-Simons theory and its $N = 2, 1$ deformations in a similar manner using branes.

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